

# LOG-SOBOLEV INEQUALITIES FOR INFINITE DIMENSIONAL GIBBS MEASURES OF HIGHER ORDER INTERACTIONS.

IOANNIS PAPAGEORGIOU

**ABSTRACT.** We focus on the infinite dimensional Log-Sobolev inequality for spin systems on the  $d$ -dimensional Lattice ( $d \geq 1$ ) with interactions of higher power than quadratic. We show that when the one dimensional single-site measure with boundaries satisfies the Log-Sobolev inequality uniformly on the boundary conditions then the infinite dimensional Gibbs measure also satisfies the inequality if the phase dominates over the interactions.

## 1. INTRODUCTION

Our focus is on the the typical Logarithmic Sobolev Inequality (LS) for measures related to systems of unbounded spins on a  $d$ -dimensional lattice, for  $d \geq 1$ , with nearest neighbour interactions of order higher than two. The aim of this paper is to investigate appropriate conditions on the local specification so that the inequality can be extended from the one site to the infinite dimensional Gibbs measure.

The main assumption is that the Log-Sobolev Inequality is true for the single site measure with a constant uniformly bound on the boundary conditions and that the power of the interaction is dominated by that of the phase.

Regarding the Log-Sobolev Inequality for the local specification  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  on a  $d$ -dimensional Lattice, criterions and examples of measures  $\mathbb{E}^{\Lambda, \omega}$  with quadratic interactions that satisfy the Log-Sobolev -with a constant uniformly on the set  $\Lambda$  and the boundary conditions  $\omega$ — are investigated in [Z2], [B], [B-E], [B-L], [Y], [A-B-C] and [B-H]. Furthermore, in [G-R] the Spectral Gap Inequality is proved. For the measure  $\mathbb{E}^{\{i\}, \omega}$  on the real line, necessary and sufficient conditions are presented in [B-G], [B-Z] and [R-Z], so that the Log-Sobolev Inequality is satisfied uniformly on the boundary conditions  $\omega$ . Furthermore, the problem of the Log-Sobolev inequality for the Infinite dimensional Gibbs measure on the Lattice is examined in [G-Z], [Z1] and [Z2]. Still in the case of bounded interactions, in [M], [I-P] and [O-R], criterions are presented in order to pass from the Log-Sobolev Inequality for the single-site measure  $\mathbb{E}^{\{i\}, \omega}$  to the LS for the Gibbs.

---

2000 *Mathematics Subject Classification.* 60E15, 26D10.

*Key words and phrases.* Logarithmic Sobolev Inequality, Gibbs measure, Spin systems.

*Address:* Department of Mathematics, Uppsala University, P.O Box 480, Uppsala 751 06, Sweden.

*Email:* ioannis.papageorgiou@math.uu.se, papyannis@yahoo.com.

Related to the current case of non quadratic interactions in [Pa2] conditions were presented for the stronger Logarithmic Sobolev  $q$  inequality for  $q \leq 2$  for spins on the one dimensional lattice. There the inequality for the infinite dimensional Gibbs measure was related to the inequality for the finite projection of the Gibbs measure.

In the current paper we focus on the typical Logarithmic Sobolev inequality ( $q = 2$ ) for spin systems on the  $d$ -dimensional lattice, for  $d \geq 1$ .

Consider the one dimensional measure

$$\mathbb{E}^{\{i\},\omega}(dx_i) = \frac{e^{-\phi(x_i) - \sum_{j \sim i} J_{ij} V(x_i, \omega_j)} dX_i}{Z^{\{i\},\omega}} \quad \text{with} \quad \|\partial_x \partial_y V(x, y)\|_\infty = \infty$$

Assume that  $\mathbb{E}^{\{i\},\omega}$  satisfies the (LS) inequality with a constant uniformly on  $\omega$ . Our aim is to set conditions, so that the infinite volume Gibbs measure  $\nu$  for the local specification  $\{\mathbb{E}^{\Lambda,\omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  satisfies the LS inequality.

Our general setting is as follows:

*The Lattice.* When we refer to the Lattice we mean the  $d$ -dimensional square Lattice  $\mathbb{Z}^d$  for  $d \geq 1$ .

*The one site space  $\mathcal{M}$ .* We consider continuous unbounded random variables in an  $n$ -dimensional space  $\mathcal{M}$ , representing spins. The space  $\mathcal{M}$ , is a  $n$ -dimensional non compact metric spaces. We will denote  $d$  the distance and  $\nabla$  the (sub)gradient for which we assume that

$$0 < \xi < |\nabla d| \leq \tau$$

for some  $\tau, \xi \in (0, \infty)$ , and

$$|\Delta d| < \theta$$

outside the unit ball  $\{d(x) < 1\}$  for some  $\theta \in (0, +\infty)$ .

When we refer to the (sub)gradient  $\nabla$  or (sub)Laplacian  $\Delta$  of  $\mathcal{M}$  related to a specific node, say  $i \in \mathbb{Z}^d$ , we will indicate this by the use of indices, i.e. we will write  $\nabla_i$  and  $\Delta_i$ .

*The Configuration space.* Our configuration space is  $\Omega = \mathcal{M}^{\mathbb{Z}^d}$ . We consider functions  $f : \Omega \rightarrow \mathbb{R}$ . Accordingly we define  $\nabla_i f(\omega) := \nabla f_i(x|\omega)|_{x=\omega_i}$  and  $\Delta_i f(\omega) := \Delta f_i(x|\omega)|_{x=\omega_i}$  for suitable  $f$ , where  $\nabla$  and  $\Delta$  are the (sub)gradient and the (sub)Laplacian on  $\mathcal{M}$  respectively. For  $\Lambda \subset \mathbb{Z}^d$ , set  $\nabla_\Lambda f = (\nabla_i f)_{i \in \Lambda}$  and

$$|\nabla_\Lambda f|^2 := \sum_{i \in \Lambda} |\nabla_i f|^2.$$

We will write  $\nabla_{\mathbb{Z}^d} = \nabla$ , since it will not cause any confusion. For any  $\omega \in \Omega$  and  $\Lambda \subset \mathbb{Z}^d$  we denote

$$\omega = (\omega_i)_{i \in \mathbb{Z}^d}, \omega_\Lambda = (\omega_i)_{i \in \Lambda}, \omega_{\Lambda^c} = (\omega_i)_{i \in \Lambda^c} \quad \text{and} \quad \omega = \omega_\Lambda \circ \omega_{\Lambda^c}$$

where  $\omega_i \in \mathcal{M}$ . When  $\Lambda = \{i\}$  we will write  $\omega_i = \omega_{\{i\}}$ . Furthermore, we will write  $i \sim j$  when the nodes  $i$  and  $j$  are nearest neighbours, that means, they are connected with a vertex, while we will denote the set of the neighbours of  $k$  as  $\{\sim k\} = \{r : r \sim k\}$ .

*The functions of the configuration.* Let  $f: \Omega \rightarrow \mathbb{R}$ . We consider integrable functions  $f$  that depend on a set of variables  $\{x_i\}, i \in \Sigma_f$  for a finite subset  $\Sigma_f \subset \mathbb{Z}^d$ . The symbol  $\subset\subset$  is used to denote a finite subset.

*The Measure on  $\mathbb{Z}^d$ .* For any subset  $\Lambda \subset\subset \mathbb{Z}$  we define the probability measure

$$\mathbb{E}^{\Lambda, \omega}(dX_\Lambda) = \frac{e^{-H^{\Lambda, \omega}} dX_\Lambda}{Z^{\Lambda, \omega}}$$

where

- $X_\Lambda = (x_i)_{i \in \Lambda}$  and  $dX_\Lambda = \prod_{i \in \Lambda} dx_i$
- $Z^{\Lambda, \omega} = \int e^{-H^{\Lambda, \omega}} dX_\Lambda$
- $H^{\Lambda, \omega} = \sum_{i \in \Lambda} \phi(x_i) + \sum_{i \in \Lambda, j \sim i} J_{ij} V(x_i, z_j)$

and

$$\bullet \quad z_j = x_\Lambda \circ \omega_{\Lambda^c} = \begin{cases} x_j & , i \in \Lambda \\ \omega_j & , i \notin \Lambda \end{cases}$$

We call  $\phi$  the phase and  $V$  the potential of the interaction. For convenience we will frequently omit the boundary symbol from the measure and we will write  $\mathbb{E}^\Lambda \equiv \mathbb{E}^{\Lambda, \omega}$ .

For the phase  $\phi$  we make the following assumptions

- **(H1.1)** there exists some  $p \geq 3$  and  $k_0 > 0$  such that

$$\partial_{d(x_i)} \phi(x_i) \geq k_0 d^{p-1}(x_i)$$

- **(H1.2)** there exists some  $k_1 > 0$  such that

$$|\partial_{d(x_i)}^2 \phi(x_i)| \leq k_1 + k_1 \partial_{d(x_i)} \phi(x_i)$$

Furthermore, for the interactions  $V(x_i, \omega_j)$  we make the following assumptions

- **(H1.3)**  $V(x, y)$  is a function of the distance  $d$  such that

$$\partial_{d(x)} V(x, y) \geq 0$$

- **(H1.4)** there exists some  $k_2 > 0$  such that

$$|\partial_{d(x_i)}^2 V(x_i, \omega_j)| \leq k_2 + k_2 \partial_{d(x_i)} V(x_i, \omega_j)$$

for any  $d(x_i) > M^*$ .

- **(H1.5)** there exist an  $0 < s \leq p-1$  such that for some  $k > 0$  and any  $j \sim i$

$$(a) \quad \mathbb{E}^{j, \omega} |\nabla_j V(x_i, \omega_j)|^2 \leq k + k \sum_{i \sim j} d^s(\omega_i)$$

$$(b) \quad \mathbb{E}^{j, \omega} d(x_j) \leq k + k \sum_{i \sim j} d^s(\omega_i)$$

$$(c) \quad |\nabla_j V(x_i, \omega_j)|^2 \leq k + k(d^s(x_i) + d^s(\omega_j))$$

$$(d) \quad |V(x_i, \omega_j)| \leq k + k(d^s(x_i) + d^s(\omega_j))$$

as well as

$$(e) \quad \mathbb{E}^{j, \omega} e^{\epsilon |\nabla_j V(x_i, \omega_j)|^2} \leq e^k e^{k \sum_{i \sim j} d^s(\omega_i)}$$

for some  $\epsilon > 0$  sufficiently small.

- **(H1.6)** The coefficients  $J_{i,j}$  are such that  $|J_{i,j}| \in [0, J]$  for some  $J < 1$  sufficiently small.

*The Infinite Volume Gibbs Measure.* The Gibbs measure  $\nu$  for the local specification  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset G, \omega \in \Omega}$  is defined as the probability measure which solves the Dobrushin-Lanford-Ruelle (DLR) equation

$$\nu \mathbb{E}^{\Lambda, \star} = \nu$$

for finite sets  $\Lambda \subset \mathbb{Z}$  (see [Pr]). For conditions on the existence and uniqueness of the Gibbs measure see e.g. [B-H.K] and [D]. In this paper we consider local specifications for which the Gibbs measure exists and it is unique. It should be noted that  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  always satisfies the DLR equation, in the sense that

$$\mathbb{E}^{\Lambda, \omega} \mathbb{E}^{M, \star} = \mathbb{E}^{\Lambda, \omega}$$

for every  $M \subset \Lambda$ . [P]

We denote

$$\mathbb{E}^{\Lambda, \omega} f = \int f d\mathbb{E}^{\Lambda, \omega}(X_\Lambda)$$

We can define the following inequalities

*The Log-Sobolev Inequality (LS).* We say that the measure  $\mathbb{E}^{\Lambda, \omega}$  satisfies the Log-Sobolev Inequality, if there exists a constant  $C_{LS}$  such that for any function  $f$ , the following holds

$$\mathbb{E}^{\Lambda, \omega} |f|^2 \log \frac{|f|^2}{\mathbb{E}^{\Lambda, \omega} |f|^2} \leq C_{LS} \mathbb{E}^{\Lambda, \omega} |\nabla_\Lambda f|^2$$

with a constant  $C_{LS} \in (0, \infty)$  uniformly on the set  $\Lambda$  and the boundary conditions  $\omega$ .

*The Spectral Gap Inequality.* We say that the measure  $\mathbb{E}^{\Lambda, \omega}$  satisfies the Spectral Gap Inequality, if there exists a constant  $C_{SG}$  such that for any function  $f$ , the following holds

$$\mathbb{E}^{\Lambda, \omega} |f - \mathbb{E}^{\Lambda, \omega} f|^2 \leq C_{SG} \mathbb{E}^{\Lambda, \omega} |\nabla_\Lambda f|^2$$

with a constant  $C_{SG} \in (0, \infty)$  uniformly on the set  $\Lambda$  and the boundary conditions  $\omega$ .

**Remark 1.1.** *We will frequently use the following two well known properties about the Log-Sobolev and the Spectral Gap Inequality. If the probability measure  $\mu$  satisfies the Log-Sobolev Inequality with constant  $c$  then it also satisfies the Spectral Gap Inequality with a constant less or equal than  $c$ . Furthermore, if for a family  $I$*

of sets  $\Lambda_i \subset \mathbb{Z}^d$ ,  $\text{dist}(\Lambda_i, \Lambda_j) > 1$ ,  $i \neq j$  the measures  $\mathbb{E}^{\Lambda_i, \omega}$ ,  $i \in I$  satisfy the Log-Sobolev Inequality with constants  $c_i$ ,  $i \in I$ , then the probability measure  $\mathbb{E}^{\{\cup_{i \in I} \Lambda_i\}, \omega}$  also satisfies the (LS) Inequality with constant  $c = \max_{i \in I} c_i$ . The last result is also true for the Spectral Gap Inequality. The proofs of these properties can be found in [G], [G-Z] and [B-Z].

## 2. THE MAIN RESULT

We want to extend the Log-Sobolev Inequality from the single-site measure  $\mathbb{E}^{\{i\}, \omega}$  to the Gibbs measure for the local specification  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  on the entire  $d$  dimensional Lattice. In the remaining of this paper we will refer to the hypothesis about the phase and the interactions (H1.0)-(H1.4) collectively as **(H1)**. The main hypothesis about the one site measure will be denoted as (H0):

**(H0):** The one dimensional measures  $\mathbb{E}^{i, \omega}$  satisfy the Log-Sobolev Inequality with a constant  $c$  uniformly with respect to the boundary conditions  $\omega$ .

Now we can state the main theorem.

**Theorem 2.1.** *Let  $f: \mathcal{M}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ . If hypothesis (H0) and (H1) for  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  are satisfied, then the infinite dimensional Gibbs measure  $\nu$  for the local specification  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  satisfies the Log-Sobolev inequality*

$$\nu |f|^2 \log \frac{|f|^2}{\nu |f|^2} \leq \mathfrak{C} \nu |\nabla f|^2$$

for some positive constant  $\mathfrak{C}$ .

The main assumption about the local specification has been that the one site measure  $\mathbb{E}^{i, \omega}$  satisfies the Log-Sobolev Inequality with a constant uniformly to the boundary conditions, while the main assumption about the interactions is that the phase  $\phi(x)$  dominates over the interactions, in the sense that

$$|\nabla_j V(x_i, \omega_j)|^2 \leq k + k(d^s(x_i) + d^s(\omega_j)) \leq k + \frac{k}{k_0}(\partial_{d(x_i)} \phi(x_i) + \partial_{d(\omega_j)} \phi(\omega_j))$$

for  $s \leq p-1$ . Then the Log-Sobolev inequality is extended to the infinite dimensional Gibbs measure. In other words, what we roughly require is for a phase of order  $d^p$  the interaction to be the most of order  $d^{\frac{p+1}{2}}$ .

As an example of a measure  $\mathbb{E}^{i, \omega}$  that satisfies (H0), that is the Log-Sobolev Inequality with a constant  $c$  uniformly with respect to the boundary conditions  $\omega$  with non quadratic interaction, one can think the following measure on the Heisenberg group

$$H^{\Lambda, \omega}(x_\Lambda) = \alpha \sum_{i \in \Lambda} d^p(x_i) + \varepsilon \sum_{\{i, j\} \cap \Lambda \neq \emptyset, j: j \sim i} (d(x_i) + \rho d(\omega_j))^s$$

for  $\alpha > 0$ ,  $\varepsilon, \rho \in \mathbb{R}$ , and  $p > s > 2$ , where as above  $x_i = \omega_i$  for  $i \notin \Lambda$ . The proof of this follows with the use of uniform U-Bounds (see [I-P]).

We briefly mention some consequences of this result. The first follows directly from Remark 1.1.

**Corollary 2.2.** *Let  $\nu$  be as in Theorem 2.1. Then  $\nu$  satisfies the Spectral Gap inequality*

$$\nu |f - \nu f|^2 \leq \mathfrak{C} \nu |\nabla f|^2$$

where  $\mathfrak{C}$  is as in Theorem 2.1.

The proofs of the next two can be found in [B-Z].

**Corollary 2.3.** *Let  $\nu$  be as in Theorem 2.1 and suppose  $f : \Omega \rightarrow \mathbb{R}$  is such that  $\|\nabla f\|_\infty < 1$ . Then*

$$\nu(e^{\lambda f}) \leq \exp\{\lambda \nu(f) + \mathfrak{C} \lambda^2\}$$

for all  $\lambda > 0$  where  $\mathfrak{C}$  is as in Theorem 2.1. Moreover, by applying Chebyshev's inequality, and optimising over  $\lambda$ , we arrive at the following 'decay of tails' estimate

$$\nu\left\{\left|f - \int f d\nu\right| \geq h\right\} \leq 2 \exp\left\{-\frac{1}{\mathfrak{C}} h^2\right\}$$

for all  $h > 0$ .

**Corollary 2.4.** *Suppose that our configuration space is actually finite dimensional, so that we replace  $\mathbb{Z}^d$  by some finite graph  $G$ , and  $\Omega = (\mathcal{M})^G$ . Then Theorem 2.1 still holds, and implies that if  $\mathcal{L}$  is a Dirichlet operator satisfying*

$$\nu(f \mathcal{L} f) = -\nu(|\nabla f|^2),$$

then the associated semigroup  $P_t = e^{t\mathcal{L}}$  is ultracontractive.

**Remark 2.5.** *In the above we are only considering interactions of range 1, but we can easily extend our results to deal with the case where the interaction is of finite range  $R$ .*

**Proof of Theorem 2.1.** We want to extend the Log-Sobolev Inequality from the single-site measure  $\mathbb{E}^{\{i\}, \omega}$  to the Gibbs measure for the local specification  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  on the entire lattice. To do so, we will follow the iterative method developed by Zegarlinski in [Z1] and [Z2] (see also [Pa2] and [I-P] for similar application). Without loss of generality in proof of the theorem we will assume that  $d = 2$ , that is, that the configuration space is  $\Omega = \mathcal{M}^{\mathbb{Z}^2}$ . Define the following sets

$$\begin{aligned} \Gamma_0 &= (0, 0) \cup \{j \in \mathbb{Z}^2 : \text{dist}(j, (0, 0)) = 2m \text{ for some } m \in \mathbb{N}\}, \\ \Gamma_1 &= \mathbb{Z}^2 \setminus \Gamma_0. \end{aligned}$$

where  $\text{dist}(i, j)$  refers to the distance of the shortest path (number of vertices) between two nodes  $i$  and  $j$ . Note that  $\text{dist}(i, j) > 1$  for all  $i, j \in \Gamma_k, k = 0, 1$  and

$\Gamma_0 \cap \Gamma_1 = \emptyset$ . Moreover  $\mathbb{Z}^2 = \Gamma_0 \cup \Gamma_1$ . As above, for the sake of notation, we will write  $\mathbb{E}_{\Gamma_k} = \mathbb{E}_{\Gamma_k}^\omega$  for  $k = 0, 1$ . Denote

$$(2.1) \quad \mathcal{P} = \mathbb{E}^{\Gamma_1} \mathbb{E}^{\Gamma_0}$$

In order to prove the Log-Sobolev Inequality for the measure  $\nu$ , we will express the entropy with respect to the measure  $\nu$  as the sum of the entropies of the measures  $\mathbb{E}^{\Gamma_0}$  and  $\mathbb{E}^{\Gamma_1}$  which are easier to handle. We can write

$$(2.2) \quad \begin{aligned} \nu(f^2 \log \frac{f^2}{\nu f^2}) &= \nu \mathbb{E}^{\Gamma_0} (f^2 \log \frac{f^2}{\mathbb{E}^{\Gamma_0} f^2}) + \nu \mathbb{E}^{\Gamma_1} (\mathbb{E}^{\Gamma_0} f^2 \log \frac{\mathbb{E}^{\Gamma_0} f^2}{\mathbb{E}^{\Gamma_1} \mathbb{E}^{\Gamma_0} f^2}) + \\ &\quad \nu(\mathbb{E}^{\Gamma_1} \mathbb{E}^{\Gamma_0} f^2 \log \mathbb{E}^{\Gamma_1} \mathbb{E}^{\Gamma_0} f^2) - \nu(f^2 \log \nu f^2) \end{aligned}$$

According to hypothesis (H0), the Log-Sobolev Inequality is satisfied for the single-state measures  $\mathbb{E}^{\{j\}}$  and the sets  $\Gamma_k$  are unions of one dimensional sets of distance greater than the length of the interaction one. Thus, as we mentioned in Remark 1.1 in the introduction, the (LS) holds for the product measures  $\mathbb{E}^{\Gamma_k}$  with the same constant  $c$ . If we use the LS for  $\mathbb{E}^{\Gamma_i}, i = 0, 1$  we get

$$(2.3) \quad \begin{aligned} (2.2) &\leq c\nu(\mathbb{E}^{\Gamma_0} |\nabla_0 f|^2) + c\nu \mathbb{E}^{\Gamma_1} \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} f^2)^{\frac{1}{2}} \right|^q + \\ &\quad \nu(\mathcal{P}^1 f^2 \log \mathcal{P}^1 f^2) - \nu(f^2 \log \nu f^2) \end{aligned}$$

For the third term of (2.3) we can write

$$\begin{aligned} \nu(\mathcal{P}^1 f^2 \log \mathcal{P}^1 f^2) &= \nu \mathbb{E}^{\Gamma_2} (\mathcal{P}^1 f^2 \log \frac{\mathcal{P}^1 f^2}{\mathbb{E}^{\Gamma_2} \mathcal{P}^1 f^2}) + \nu \mathbb{E}^{\Gamma_3} (\mathcal{P}^2 f^2 \log \frac{\mathcal{P}^2 f^2}{\mathbb{E}^{\Gamma_3} \mathcal{P}^2 f^2}) + \\ &\quad \nu(\mathbb{E}^{\Gamma_3} \mathcal{P}^2 f^2 \log \mathbb{E}^{\Gamma_3} \mathcal{P}^2 f^2) \end{aligned}$$

If we use again the Log-Sobolev Inequality for the measures  $\mathbb{E}^{\Gamma_i}, i = 2, 3$  we get

$$(2.4) \quad \nu(\mathcal{P}^1 f^2 \log \mathcal{P}^1 f^2) \leq c\nu \left| \nabla_{\Gamma_2} (\mathcal{P}^1 f^2)^{\frac{1}{2}} \right|^q + c\nu \left| \nabla_{\Gamma_3} (\mathcal{P}^2 f^2)^{\frac{1}{2}} \right|^2 + \nu(\mathcal{P}^3 f^2 \log \mathcal{P}^3 f^2)$$

If we work similarly for the last term  $\nu(\mathcal{P}^3 f^2 \log \mathcal{P}^3 f^2)$  of (2.4) and inductively for any term  $\nu(\mathcal{P}^k f^2 \log \mathcal{P}^k f^2)$ , then after  $n$  steps (2.3) and (2.4) will give

$$(2.5) \quad \begin{aligned} \nu(f^2 \log \frac{f^2}{\nu f^2}) &\leq \nu(\mathcal{P}^n f^2 \log \mathcal{P}^n f^2) - \nu(f^2 \log \nu f^2) + \\ &\quad c\nu |\nabla_0 f|^2 + c \sum_{k=1}^n \nu \left| \nabla_{\Gamma_k} (\mathcal{P}^{k-1} f^2)^{\frac{1}{2}} \right|^2 \end{aligned}$$

In order to calculate the third and fourth term on the right-hand side of (2.5) we will use the following proposition

**Proposition 2.6.** *Suppose that hypothesis (H0) and (H1) are satisfied. Then the following bound holds*

$$(2.6) \quad \nu \left| \nabla_{\Gamma_i} (\mathbb{E}^{\Gamma_j} f^2)^{\frac{1}{2}} \right|^2 \leq C_1 \nu |\nabla_{\Gamma_i} f|^2 + C_2 \nu |\nabla_{\Gamma_j} f|^2$$

for  $\{i, j\} = \{0, 1\}$  and constants  $C_1 \in (0, \infty)$  and  $0 < C_2 < 1$ .

The proof of Proposition 2.6 will be the subject of Section 4. If we apply inductively relationship (2.6)  $k$  times to the third and the fourth term of (2.5) we obtain

$$(2.7) \quad \nu \left| \nabla_{\Gamma_0} (\mathcal{P}^k f^2)^{\frac{1}{2}} \right|^2 \leq C_2^{2k-1} C_1 \nu |\nabla_{\Gamma_1} f|^2 + C_2^{2k} \nu |\nabla_{\Gamma_0} f|^2$$

and

$$(2.8) \quad \nu \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} \mathcal{P}^k f^2)^{\frac{1}{2}} \right|^2 \leq C_2^{2k} C_1 \nu |\nabla_{\Gamma_1} f|^2 + C_2^{2k+1} \nu |\nabla_{\Gamma_0} f|^2$$

If we plug (2.7) and (2.8) in (2.5) we get

$$(2.9) \quad \begin{aligned} \nu(f^2 \log \frac{f^2}{\nu f^2}) &\leq \nu(\mathcal{P}^n f^2 \log \mathcal{P}^n f^2) - \nu(f^2 \log \nu f^2) + \\ &\quad c \left( \sum_{k=0}^{n-1} C_2^{2k-1} \right) C_1 \nu |\nabla_{\Gamma_1} f|^2 + c \left( \sum_{k=0}^{n-1} C_2^{2k} \right) \nu |\nabla_{\Gamma_0} f|^2 + \\ &\quad c \left( \sum_{k=0}^{n-1} C_2^{2k} \right) C_1 \nu |\nabla_{\Gamma_1} f|^2 + c \left( \sum_{k=0}^{n-1} C_2^{2k+1} \right) \nu |\nabla_{\Gamma_0} f|^2 \end{aligned}$$

If we take the limit of  $n$  to infinity in (2.9) the first two terms on the right hand side cancel with each other, as explained on the proposition bellow.

**Proposition 2.7.** *Under hypothesis (H0) and (H1),  $\mathcal{P}^n f$  converges  $\nu$ -almost everywhere to  $\nu f$ .*

The proof of this proposition will be presented in Section 3. So, taking the limit of  $n$  to infinity in (2.9) leads to

$$\nu(|f|^2 \log \frac{|f|^2}{\nu |f|^2}) \leq cA \left( \frac{C_1}{C_2} + C_2 + C_1 \right) \nu |\nabla_{\Gamma_1} f|^2 + cA \nu |\nabla_{\Gamma_0} f|^2$$

where  $A = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} C_2^{2k} < \infty$  for  $C_2 < 1$ , and the theorem follows for a constant  $C = \max\{cA \left( \frac{C_1}{C_2} + C_2 + C_1 \right), cA\}$   $\square$

### 3. PROOF OF PROPOSITION 2.7.

Before proving Proposition 2.7 we will present the key proposition of this paper, Proposition 3.2. This proposition will also be used in the next section 4 where Proposition 2.6 is proved.

In the case of quadratic interactions  $V(x, y) = (x - y)^2$  one can calculate

$$\mathbb{E}^{i, \omega} (f^2 (\nabla_j V(x_i - x_j) - \mathbb{E}^{i, \omega} \nabla_j V(x_i - x_j))^2)$$

(see [B-H] and [H]) with the use of the relative entropy inequality (see [D-S]) and the Herbst argument (see [L] and [H]). Herbst's argument states that if a



probability measure  $\mu$  satisfies the LS inequality and a function  $F$  is Lipschitz continues with  $\|F\|_{Lips} \leq 1$  and such that  $\mu(F) = 0$ , then for some small  $\epsilon$  we have

$$\mu e^{\epsilon F^2} < \infty$$

For  $\mu = \mathbb{E}^{i,\omega}$  and  $F = \frac{\nabla_j V(x_i - x_j) - \mathbb{E}^{i,\omega} \nabla_j V(x_i - x_j)}{2}$  we then obtain

$$\mathbb{E}^{i,\omega} e^{\frac{\epsilon}{4} (\nabla_j V(x_i - x_j) - \mathbb{E}^{i,\omega} \nabla_j V(x_i - x_j))^2} < \infty$$

uniformly on the boundary conditions  $\omega$ , because of hypothesis (H0). In the more general case however of non quadratic interactions that we examine in this work, the Herbst argument cannot be applied. In this and next sections we show how one can handle exponential quantities like the last one with the use of U-bound inequalities and hypothesis (H1) for the interactions.

U-bound inequalities introduced in [H-Z] are used to prove Logarithmic Sobolev and Logarithmic Sobolev type inequalities. In the aforementioned paper, the U-bound inequalities

$$(3.1) \quad \mu(f^q d^r) \leq C\mu|\nabla f|^q + D\mu|f|^q$$

where used to prove Log-Sobolev  $q$  inequalities for  $q \in (1, 2]$ , the spectral Gap inequality, as well as F-Sobolev inequalities. In particular, the U-bound inequality (3.1) for  $q \in (1, 2]$  and  $r$  bigger than the conjugate of  $q$  was used to prove that the measure  $e^{-d^r(x)} dx / \int e^{-d^r(x)} dx$  satisfies the Log-Sobolev  $q$  inequality. In the context of the typical Log-Sobolev inequality, this implies that the measure

$$\frac{e^{-d^r(x)} dx}{\int e^{-d^r(x)} dx}$$

for  $r > 2$  satisfies the Log-Sobolev inequality. In [I-P], the U-Bound inequality (3.1) was shown for the one site measure  $\mathbb{E}^{i,\omega}$  uniformly on the boundary conditions for a specific example of a measure on the Heisenberg group with quadratic interactions. However, it appears that this is very difficult to obtain in general when boundary conditions are involved. In this paper however, where the Log-Sobolev inequality is assumed for the single node measure  $\mathbb{E}^{i,\omega}$  uniformly on the boundary conditions, as stated in hypothesis (H0), the strong U-Bound inequality (3.1) for  $r > 2$  and  $q = 2$  is not necessary. Instead we will prove the weaker version of it

$$(3.2) \quad \mathbb{E}^{i,\omega} d^s(x_i) f^2 \leq C \mathbb{E}^{i,\omega} |\nabla_i f|^2 + C \mathbb{E}^{i,\omega} f^2$$

for  $s < p - 1$ . This will be then used in order to control the interactions and prove sweeping out relations as in Proposition 2.6 and Lemma 3.3. Weaker U-Bound inequalities than (3.1) have been used in the past for measures without interaction in [H-Z] and [Pa1] in order to prove weaker inequalities than the Log-Sobolev, like F-Sobolev and Modified Log-Sobolev inequalities. In effect, we focus on bounding  $\mathbb{E}^{i,\omega} d^s(x_i) f^2$  instead of  $\mathbb{E}^{i,\omega} d^p(x_i) f^2$  that would had been the appropriate analogue U-bound for the Log-Sobolev inequality, than the inferior (3.2), since it contains constants independent of the boundary conditions. Furthermore for  $p$  larger than

the interactions power we can control the boundary conditions. Following closely on the proof of U-bound inequalities for the free boundary measure in [H-Z], the main U-bound inequality is proven in the following proposition.

Denote

$$D^{i,\omega} = \partial_{d(x_i)}\phi(x_i) + \sum_{j \sim i} J_{ij} \partial_{d(x_i)} V(x_i, \omega_j)$$

and

$$B^{i,\omega} = \partial_{d(x_i)}^2 \phi(x_i) + \sum_{j \sim i} J_{ij} \partial_{d(x_i)}^2 V(x_i, \omega_j)$$

**Lemma 3.1.** *For any  $\omega_j, j \sim i$ , denote  $B_N$  a set such that for all  $x_i \in B_N$  to have*

$$\sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) < N$$

*There exist large enough constants  $M > 0$  and  $N > 0$  such that for every  $x_i \in \mathcal{O} := \{\partial_{d(x_i)}\phi(x_i) > M\} \cup \{\{\partial_{d(x_i)}\phi(x_i) < M\} \cap B_N^c\}$  the following to hold*

$$|D^{i,\omega}|(\Delta_i d) + B^{i,\omega} |\nabla_i d|^2 \leq \frac{\zeta}{2} |D^{i,\omega}|^2 |\nabla_i d|^2$$

for some  $\zeta < 1$ .

*Proof.* To prove the lemma we will distinguish two main cases. At first we assume  $x_i : \partial_{d(x_i)}\phi(x_i) > M$ . In this case we will consider two sub cases, (a)  $x_i \in B_N^c$  and (b)  $x_i \in B_N$ . Then we will examine the third case (c)  $x_i \in \{\partial_{d(x_i)}\phi(x_i) < M\} \cap B_N^c$ .

a)  $x_i \in \{\partial_{d(x_i)}\phi(x_i) > M\} \cap B_N^c$ .

Because of (H1.2) and (H1.4) we get

$$B^{i,\omega} \leq k_1 + k_1 \partial_{d(x_i)}\phi(x_i) + k_2 + k_2 \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j)$$

We can then compute

$$(3.3) \quad \begin{aligned} D^{i,\omega}(\Delta_i d) + B^{i,\omega} |\nabla_i d|^2 &< (k_1 + k_2)\tau^2 + (\theta + k_1\tau^2)\partial_{d(x_i)}\phi \\ &+ (\theta + k_2\tau^2) \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \end{aligned}$$

We also have

$$(3.4) \quad \frac{1}{2} |D^{i,\omega}|^2 |\nabla_i d|^2 \geq \frac{\xi^2}{2} \left( \partial_{d(x_i)}\phi + \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \right)^2$$

where above we used the fact that

$$0 < \xi < |\nabla d| \leq \tau$$

for some  $\tau, \xi \in (0, \infty)$ , and

$$|\Delta d| < \theta$$

outside the unit ball  $\{d(x) < 1\}$  for some  $\theta \in (0, +\infty)$ . If we choose  $M$  and  $N$  large enough such that

$$\begin{aligned} \frac{\zeta_1 \xi^2}{2} \left( \partial_{d(x_i)} \phi + \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \right)^2 &> (k_1 + k_2) \tau^2 + (\theta + k_1 \tau^2) \partial_{d(x_i)} \phi + \\ &(\theta + k_2 \tau^2) \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \end{aligned}$$

for some  $0 < \zeta_1 < 1$ , then from (3.3) and (3.4) we obtain

$$(3.5) \quad \frac{\zeta_1}{2} |D^{i,\omega}|^2 |\nabla_i d|^2 \geq |D^{i,\omega}| (\Delta_i d) + B^{i,\omega} |\nabla_i d|^2$$

for  $0 < \zeta_1 < 1$ .

b)  $x_i \in \{\partial_{d(x_i)} \phi > M\} \cap B_N$ .

In this case, according to (H1.4) we have

$$\left| \sum_{j \sim i} J_{i,j} \partial_{d(x_i)}^2 V(x_i, \omega_j) \right| \leq k_2 + k_2 \sum_{j \sim i} J_{i,j} \partial_{d(x_i)}^2 V(x_i, \omega_j) \leq k_2 + k_2 N$$

which together with (H1.2) gives

$$B^{i,\omega} \leq k_1 + k_1 \partial_{d(x_i)} \phi + k_2 + k_2 N$$

Similarly

$$D^{i,\omega} \leq \partial_{d(x_i)} \phi + N$$

Combining the last two together gives

$$(3.6) \quad D^{i,\omega} (\Delta_i d) + B^{i,\omega} |\nabla_i d|^2 < (k_1 \tau^2 + \theta) \partial_{d(x_i)} \phi + \tau^2 (k_1 + k_2) + (\theta + k_2 \tau^2) N$$

If we choose  $M$  sufficiently large such that for every  $x : \partial_{d(x_i)} \phi(x) > M$  to have

$$(k_1 \tau^2 + \theta) \partial_{d(x_i)} \phi + \tau^2 (k_1 + k_2) + (\theta + k_2 \tau^2) N < \frac{\zeta_2}{2} \xi^2 (\partial_{d(x_i)} \phi)^2$$

for some  $0 < \zeta_2 < 1$ , then (3.6) becomes

$$(3.7) \quad |D^{i,\omega}| (\Delta_i d) + B^{i,\omega} |\nabla_i d|^2 < \frac{\zeta_2}{2} \xi^2 (\partial_{d(x_i)} \phi)^2 \leq \frac{\zeta_2}{2} |\nabla_i d|^2 |D^{i,\omega}|^2$$

where above we used that from (H1.3) condition  $\partial_{d(x)} V(x, y) \geq 0$ , one has  $(\partial_{d(x_i)} \phi)^2 \leq |D^{i,\omega}|^2$  and  $|\nabla_i d| \geq \xi$ .

We will now look at the last case.

c)  $x_i \in \{\partial_{d(x_i)} \phi < M\} \cap B_N^c$ .

We have

$$D^{i,\omega} \leq M + \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j)$$

and

$$\begin{aligned} B^{i,\omega} &\leq k_1 + k_1 \partial_{d(x_i)} \phi + k_2 + k_2 \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \\ &\leq k_1 + k_1 M + k_2 + k_2 \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \end{aligned}$$

where above we used once more (H1.2) and (H1.4). Combining the last two inequalities leads to

$$(3.8) \quad D^{i,\omega} (\Delta_i d) + B^{i,\omega} |\nabla_i d|^2 \leq (\theta + k_1 \tau^2) M + (k_2 + k_1) \tau^2 + (\theta + k_2 \tau^2) \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j)$$

We can also calculate

$$(3.9) \quad \frac{1}{2} |D^{i,\omega}|^2 |\nabla_i d|^2 \geq \frac{1}{2} \left| \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \right|^2 \xi^2$$

due to (H1.1). If we choose  $N$  sufficiently large so that

$$\begin{aligned} (\theta + k_1 \tau^2) M + (k_2 + k_1) \tau^2 + (\theta + k_2 \tau^2) \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) &\leq \\ \frac{\zeta_3}{2} \left| \sum_{j \sim i} J_{i,j} \partial_{d(x_i)} V(x_i, \omega_j) \right|^2 \xi^2 \end{aligned}$$

then (3.8) and (3.9) gives

$$(3.10) \quad (|D^{i,\omega}| (\Delta_i d) + B^{i,\omega} |\nabla_i d|^2) \leq \frac{\zeta_3}{2} |D^{i,\omega}|^2 |\nabla_i d|^2$$

for some  $0 < \zeta_3 < 1$ .

The lemma follows from (3.5), (3.7) and (3.10) for  $\zeta = \min\{\zeta_1, \zeta_2, \zeta_3\}$  □

**Proposition 3.2.** *For  $\mathbb{E}^{i,\omega}$  as in (H1) there exists a  $C > 0$  such that*

$$\mathbb{E}^{i,\omega} d^r f^2 \leq C \mathbb{E}^{i,\omega} |\nabla_i f|^2 + C \mathbb{E}^{i,\omega} f^2$$

for every  $r \leq 2(p-1)$ .

*Proof.* If we use the Leibnitz rule we have

$$(\nabla_i f) e^{-\frac{H^{i,\omega}}{2}} = \nabla_i (f e^{-\frac{H^{i,\omega}}{2}}) + \frac{1}{2} (\nabla_i H^{i,\omega}) f e^{-\frac{H^{i,\omega}}{2}}$$

If we square the last one and then integrate with respect to  $dx_i$  we obtain

$$(3.11) \quad \begin{aligned} \int |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i &\geq \frac{1}{4} \int |\nabla_i H^{i,\omega}|^2 f^2 e^{-H^{i,\omega}} dx_i + \\ &\int \nabla_i H^{i,\omega} \cdot \nabla_i (f e^{-\frac{H^{i,\omega}}{2}}) f e^{-\frac{H^{i,\omega}}{2}} dx_i \end{aligned}$$

But

$$\nabla_i H^{i,\omega} = \partial_{d(x_i)} \phi(x_i) \nabla_i d + \sum_{j \sim i} J_{ij} \partial_{d(x_i)} V(x_i, \omega_j) \nabla_i d = D^{i,\omega} \nabla_i d$$

Plugging the last one in (3.11) gives

$$(3.12) \quad \int |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i \geq \frac{1}{4} \int |D^{i,\omega}|^2 |\nabla_i d|^2 f^2 e^{-H^{i,\omega}} dx_i + \int D^{i,\omega} \nabla_i d \cdot \nabla_i (f e^{-\frac{H^{i,\omega}}{2}}) f e^{-\frac{H^{i,\omega}}{2}} dx_i$$

Integrating by parts in the second term on the right hand side we have

$$\begin{aligned} \int D^{i,\omega} \nabla_i d \cdot \nabla_i (f e^{-\frac{H^{i,\omega}}{2}}) f e^{-\frac{H^{i,\omega}}{2}} dx_i &= - \int (\nabla_i D^{i,\omega}) \cdot (\nabla_i d) f^2 e^{-H^{i,\omega}} dx_i - \\ &\quad \int D^{i,\omega} (\Delta_i d) f^2 e^{-H^{i,\omega}} dx_i - \int D^{i,\omega} \nabla_i d \cdot \nabla_i (f e^{-\frac{H^{i,\omega}}{2}}) f e^{-\frac{H^{i,\omega}}{2}} dx_i \end{aligned}$$

or equivalently

$$(3.13) \quad \begin{aligned} \int D^{i,\omega} \nabla_i d \cdot \nabla_i (f e^{-\frac{H^{i,\omega}}{2}}) f e^{-\frac{H^{i,\omega}}{2}} dx_i &= - \frac{1}{2} \int (\nabla_i D^{i,\omega}) \cdot (\nabla_i d) f^2 e^{-H^{i,\omega}} dx_i - \\ &\quad \frac{1}{2} \int D^{i,\omega} (\Delta_i d) f^2 e^{-H^{i,\omega}} dx_i \end{aligned}$$

From (3.12) and (3.13) we get

$$\begin{aligned} \int |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i &\geq \frac{1}{4} \int |D^{i,\omega}|^2 |\nabla_i d|^2 f^2 e^{-H^{i,\omega}} dx_i - \\ &\quad \frac{1}{2} \int D^{i,\omega} (\Delta_i d) f^2 e^{-H^{i,\omega}} dx_i - \frac{1}{2} \int (\nabla_i D^{i,\omega}) \cdot (\nabla_i d) f^2 e^{-H^{i,\omega}} dx_i \end{aligned}$$

If we write

$$\nabla_i D^{i,\omega} = B^{i,\omega} \nabla_i d$$

where we have denoted

$$B^{i,\omega} = \partial_{d(x_i)}^2 \phi(x_i) + \sum_{j \sim i} J_{ij} \partial_{d(x_i)}^2 V(x_i, \omega_j)$$

we then have

$$(3.14) \quad \begin{aligned} \int |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i &\geq \frac{1}{4} \int |D^{i,\omega}|^2 |\nabla_i d|^2 f^2 e^{-H^{i,\omega}} dx_i - \\ &\quad \frac{1}{2} \int (D^{i,\omega} (\Delta_i d) + B^{i,\omega} |\nabla_i d|^2) f^2 e^{-H^{i,\omega}} dx_i \end{aligned}$$

To continue we will distinguish two cases. At first we consider  $x_i \in \{\partial_{d(x_i)} \phi(x_i) < M\} \cap B_N$ . We then have

$$\frac{1}{2} (|D^{i,\omega}| (\Delta_i d) + B^{i,\omega} |\nabla_i d|^2) \leq \frac{1}{2} ((M + N)\theta + (k_1 + k_1 M + N) \tau^2) = \ddot{C}$$

from which (3.14) becomes

$$(3.15) \quad \int_{\{\partial_{d(x_i)}\phi(x_i) < M\} \cap B_N} |D^{i,\omega}|^2 |\nabla_i d|^2 f^2 e^{-H^{i,\omega}} dx_i \leq 2 \int_{\{\partial_{d(x_i)}\phi(x_i) < M\} \cap B_N} |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i + 2\ddot{C} \int_{\{\partial_{d(x_i)}\phi(x_i) < M\} \cap B_N} f^2 e^{-H^{i,\omega}} dx_i$$

In the case where  $x_i$  belongs in the complement of  $\{\partial_{d(x_i)}\phi(x_i) < M\} \cap B_N$ , that is  $x_i \in \mathcal{O} := \{\partial_{d(x_i)}\phi(x_i) > M\} \cup \{\{\partial_{d(x_i)}\phi(x_i) < M\} \cap B_N^c\}$  we can use Lemma 3.1 for  $N$  and  $M$  sufficiently large, to bound the right hand side of (3.14). We then get

$$(3.16) \quad \int_{\mathcal{O}} |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i \geq \frac{1-\zeta}{4} \int_{\mathcal{O}} |D^{i,\omega}|^2 |\nabla_i d|^2 f^2 e^{-H^{i,\omega}} dx_i$$

for some  $\zeta < 1$ . Combining together (3.15) and (3.16) we obtain

$$\int |D^{i,\omega}|^2 |\nabla_i d|^2 f^2 e^{-H^{i,\omega}} dx_i \leq \left(2 + \frac{1-\zeta}{4}\right) \int |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i + 2\ddot{C} \int f^2 e^{-H^{i,\omega}} dx_i$$

If we observe that

$$\xi^2 \int |D^{i,\omega}|^2 f^2 e^{-H^{i,\omega}} dx_i \leq \int |D^{i,\omega}|^2 |\nabla_i d|^2 f^2 e^{-H^{i,\omega}} dx_i$$

we then get

$$(3.17) \quad \int |D^{i,\omega}|^2 f^2 e^{-H^{i,\omega}} dx_i \leq \frac{1}{\xi^2} \left(2 + \frac{1-\zeta}{4}\right) \int |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i + \frac{2\ddot{C}}{\xi^2} \int f^2 e^{-H^{i,\omega}} dx_i$$

Furthermore, because of (H1.1) and (H1.3) we have

$$d^{2(p-1)}(x_i) \leq \frac{1}{k_0^2} (\partial_{d(x_i)}\phi(x_i))^2 \leq \frac{1}{k_0^2} |D^{i,\omega}|^2$$

If we combine together the last one with (3.17) we obtain

$$\int d^{2(p-1)}(x_i) f^2 e^{-H^{i,\omega}} dx_i \leq \frac{1}{\xi^2 k_0^2} \left(2 + \frac{1-\zeta}{4}\right) \int |\nabla_i f|^2 e^{-H^{i,\omega}} dx_i + \frac{2\ddot{C}}{k_0^2 \xi^2} \int f^2 e^{-H^{i,\omega}} dx_i$$

□

The first sweeping out relationships inequality follows.

**Lemma 3.3.** *Suppose that hypothesis (H0) and (H1) are satisfied. Then for  $j \sim i$*

$$\nu |\nabla_j(\mathbb{E}^i f)|^2 \leq D_1 \nu |\nabla_j f|^2 + D_2 \nu |\nabla_i f|^2$$

*holds for constants  $D_1 \in (0, \infty)$  and  $0 < D_2 < 1$ .*

*Proof.* If we denote  $\rho_i = \frac{e^{-H(x_i)}}{\int e^{-H(x_i)} dx}$  the density of the measure  $\mathbb{E}^i$  we can then write

$$(3.18) \quad |\nabla_j(\mathbb{E}^i f)|^2 \leq \left| \nabla_j \left( \int \rho_i f dx_i \right) \right|^2 \leq 2 \left| \int (\nabla_j f) \rho_i dx_i \right|^2 + 2\nu \left| \int \int f (\nabla_j \rho_i) dx_i \right|^2 \leq$$

$$(3.19) \quad c_1 |\mathbb{E}^i(\nabla_j f)|^2 + c_1 J^2 \nu |\mathbb{E}^i(f; \nabla_j V(x_j, x_i))|^2$$

where in (3.19) we used hypothesis (H1.4) to bound the coefficients  $J_{i,j}$  and we have denoted  $c_1 = 2^8$ . If we apply the Hölder Inequality to the first term of (3.19) we obtain

$$\begin{aligned} \nu |\nabla_j(\mathbb{E}^i f)|^2 &\leq c_1 \nu |\mathbb{E}^i(\nabla_j f)|^2 + c_1 J^2 \nu |\mathbb{E}^i(f; \nabla_j V(x_j, x_i))|^2 \leq \\ &c_1 \nu |\mathbb{E}^i(\nabla_j f)|^2 + c_1 J^2 \nu \mathbb{E}^i((f - \mathbb{E}^i f)^2 (\nabla_j V(x_j, x_i))^2) \leq \\ &c_1 \nu |\nabla_j f|^2 + k c_1 J^2 \nu \mathbb{E}^i((f - \mathbb{E}^i f)^2) + k c_1 J^2 \nu \mathbb{E}^i((f - \mathbb{E}^i f)^2 (d^s(x_j) + d^s(x_i))) \end{aligned}$$

from hypothesis (H1.3). Since according to (H0) the measures  $\mathbb{E}^{i,\omega}$  satisfy the Log-Sobolev Inequality, then as explained in Remark 1.1 they also satisfy the Spectral Gap inequality with constant  $c$  independently of the boundary conditions. If we use the Spectral Gap inequality for the measure  $\mathbb{E}^i$  to bound the second term on the right hand side we get

$$(3.20) \quad \begin{aligned} \nu |\nabla_j(\mathbb{E}^i f)|^2 &\leq c_1 \nu |\nabla_j f|^2 + k c c_1 J^2 \nu |\nabla_j f|^2 + \\ &k c_1 J^2 \nu \mathbb{E}^i((f - \mathbb{E}^i f)^2 d^s(x_i)) + k c_1 J^2 \nu \mathbb{E}^j((f - \mathbb{E}^i f)^2 d^s(x_j)) \end{aligned}$$

In order to bound the first term on the right hand side of (3.20) we can use Proposition 3.2

$$(3.21) \quad \nu \mathbb{E}^i((f - \mathbb{E}^i f)^2 d^s(x_i)) \leq C \nu |\nabla_i f|^2 + C \nu \mathbb{E}^i(f - \mathbb{E}^i f)^2 \leq C(1 + c) \nu |\nabla_i f|^2$$

where in the last one we used the Spectral Gap inequality. For the second term on the right hand side of (3.20) we can use again Proposition 3.2

$$(3.22) \quad \begin{aligned} \nu \mathbb{E}^j((f - \mathbb{E}^i f)^2 d^s(x_j)) &\leq C \nu |\nabla_j(f - \mathbb{E}^i f)|^2 + C \nu \mathbb{E}^j(f - \mathbb{E}^i f)^2 \leq \\ &2C \nu |\nabla_j f|^2 + 2C \nu |\nabla_j(\mathbb{E}^i f)|^2 + C c \nu |\nabla_i f|^2 \end{aligned}$$

where above we used again the Spectral Gap inequality for the measure  $\mathbb{E}^i$ . If we combine together (3.20), (3.21) and (3.22) we get

$$(3.23) \quad \nu |\nabla_j(\mathbb{E}^i f)|^2 \leq (c_1 J^2 2C + c_1 + k c c_1 J^2) \nu |\nabla_j f|^2 + (k c_1 J^2 C(1+c) + k c_1 J^2 C c) \nu |\nabla_i f|^2 + k c_1 J^2 2C \nu |\nabla_j(\mathbb{E}^i f)|^2$$

If we choose  $J$  sufficiently small so that

$$\frac{(c_1 C(1+c) + c_1 C c) J^2}{1 - k c_1 J^2 2C} \leq J$$

we get the lemma.  $\square$

Furthermore from (3.22) and Lemma 3.3 we also get the following Corollary.

**Corollary 3.4.** *Suppose that hypothesis (H0) and (H1) are satisfied. Then for  $i \sim j$  the following holds*

$$\nu \mathbb{E}^j((f - \mathbb{E}^i f)^2 d^s(x_j)) \leq D_3 \nu |\nabla_j f|^2 + D_3 \nu |\nabla_i f|^2$$

for some constant  $D_3 > 0$ .

**Proposition 3.5.** *Suppose that hypothesis (H0) and (H1) are satisfied. Then for  $j \sim i$*

$$\nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^2 \leq R_1 \nu |\nabla_{\Gamma_1} f|^2 + R_2 \nu |\nabla_{\Gamma_0} f|^2$$

for constants  $R_1 \in (0, \infty)$  and  $0 < R_2 < 1$ .

*Proof.*

$$(3.24) \quad \nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^2 = \sum_{i \in \Gamma_1} \nu |\nabla_i(\mathbb{E}^{\Gamma_0} f)|^2 \leq \sum_{i \in \Gamma_1} \nu |\nabla_i(\mathbb{E}^{\{\sim i\}} f)|^2$$

If we denote the neighbours of node  $i$  as  $\{\sim\} = \{j_1, j_2, j_3, j_4\}$  then for any  $i \in \Gamma_1$  we have

$$\nu |\nabla_i(\mathbb{E}^{\{\sim i\}} f)|^2 = \nu |\nabla_i(\mathbb{E}^{\{j_1, j_2, j_3, j_4\}} f)|^2 = \nu |\nabla_i(\mathbb{E}^{j_1} \mathbb{E}^{\{j_2, j_3, j_4\}} f)|^2$$

If we use Lemma 3.3 we get

$$\begin{aligned} \nu |\nabla_i(\mathbb{E}^{j_1} \mathbb{E}^{\{j_2, j_3, j_4\}} f)|^2 &\leq D_1 \nu |\nabla_i \mathbb{E}^{\{j_2, j_3, j_4\}} f|^2 + D_2 \nu |\nabla_{j_1} \mathbb{E}^{\{j_2, j_3, j_4\}} f|^2 = \\ &D_1 \nu |\nabla_i \mathbb{E}^{j_2} \mathbb{E}^{\{j_3, j_4\}} f|^2 + D_2 \nu |\nabla_{j_1} f|^2 \end{aligned}$$

since  $j_k, j_l \in \{\sim i\}$  for  $k \neq l$  have distance greater than the interaction one. If we apply again Lemma 3.3 three more times on the first term on the right hand side



of the last inequality we will finally obtain

$$\begin{aligned} \nu |\nabla_i(\mathbb{E}^{\{\sim i\}} f)|^2 &\leq D_1 \nu |\nabla_i \mathbb{E}^{\{j_2, j_3, j_4\}} f|^2 + D_2 \nu |\nabla_{j_1} \mathbb{E}^{\{j_2, j_3, j_4\}} f|^2 = \\ &D_1^3 \nu |\nabla_i f|^2 + D_1^3 D_2 \nu |\nabla_{j_4} f|^2 + D_1^2 D_2 \nu |\nabla_{j_3} f|^2 + \\ &D_1 D_2 \nu |\nabla_{j_2} f|^2 + D_2 \nu |\nabla_{j_1} f|^2 \leq \\ &D_1^3 \nu |\nabla_i f|^2 + D_1^3 D_2 \sum_{j \sim i} \nu |\nabla_j f|^2 \end{aligned}$$

Plugging the last one in (3.24) gives

$$\nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^2 = D_1^3 \sum_{i \in \Gamma_1} \nu |\nabla_i f|^2 q + 4D_1^3 D_2 \sum_{j \in \Gamma_0} \nu |\nabla_j f|^2$$

which proves the proposition for  $J$  sufficiently small so that  $4D_1^3 D_2 < 1$ .  $\square$

Now we can prove the a.e. convergence of  $\mathcal{P}^n$  to the infinite dimensional Gibbs measure  $\nu$  stated in Proposition 2.7.

**Proof of Proposition 2.7.** Following [G-Z] we will show that in  $L_1(\nu)$  we have  $\lim_{n \rightarrow \infty} \mathcal{P}^n = \nu$ . We have that

$$\begin{aligned} \nu |f - \mathbb{E}^{\Gamma_1} \mathbb{E}^{\Gamma_0} f|^2 &\leq 2^2 \nu \mathbb{E}^{\Gamma_0} |f - \mathbb{E}^{\Gamma_0} f|^2 + 2^2 \nu \mathbb{E}^{\Gamma_1} |\mathbb{E}^{\Gamma_0} f - \mathbb{E}^{\Gamma_1} \mathbb{E}^{\Gamma_0} f|^2 \\ (3.25) \quad &\leq 2^2 c \nu |\nabla_{\Gamma_0} f|^2 + 2^2 c \nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^2 \end{aligned}$$

The last inequality due to the fact that both the measures  $\mathbb{E}^{\Gamma_0}$  and  $\mathbb{E}^{\Gamma_1}$  satisfy the Log-Sobolev Inequality and the Spectral Gap inequality with constant  $c$  independently of the boundary conditions. If we use Proposition 3.5 we get

$$(3.25) \leq 2^2 c \nu |\nabla_{\Gamma_0} f|^2 + 2^2 c (R_1 \nu |\nabla_{\Gamma_1} f|^2 + R_2 \nu |\nabla_{\Gamma_0} f|^2)$$

From the last inequality we obtain that for any  $n \in \mathbb{N}$ ,

$$\nu |\mathcal{P}^n f - \mathcal{P}^{n+1} f|^2 \leq 2^2 c \nu |\nabla_{\Gamma_0} \mathcal{P}^n f|^2 + 2^2 c R_2 \nu |\nabla_{\Gamma_0} \mathcal{P}^n f|^2$$

If we use once more Lemma 3.3 we have the following bound

$$\nu |\mathcal{P}^n f - \mathcal{P}^{n+1} f|^2 \leq 2^2 c (1 + R_2) (R_1 R_2^{2n-1} \nu |\nabla_{\Gamma_1} f|^2 + R_2^{2n} \nu |\nabla_{\Gamma_0} f|^2)$$

which converges to zero as  $n$  goes to infinity, because of  $R_2 < 1$ . Thus, the sequence

$$\{\mathcal{P}^n f - \nu \mathcal{P}^n f\}_{n \in \mathbb{N}}$$

converges  $\nu$ -almost surely by the Borel-Cantelli lemma. The limit of  $\mathcal{P}^n f - \nu \mathcal{P}^n f = \mathcal{P}^n f - \nu f$  is therefore constant and hence identical to zero a.e.  $\square$

## 4. PROOF OF PROPOSITION 2.6

Before we prove Proposition 2.6 we present some useful lemmata. First we define

$$W_i^j = \nabla_i V(x_j, x_i) \quad \text{and} \quad U_i^j = |W_i^j|^2 + \mathbb{E}^j |W_i^j|^2$$

for  $j \sim i$ . Define now the quantity

$$A_i^j(f) = (\mathbb{E}^j |f|^q)^{-1} |\mathbb{E}^j(f^2; W_i^j)|^2$$

**Lemma 4.1.** *For every  $i \sim j$  the following inequality holds*

$$\left| \nabla_i (\mathbb{E}^j |f|^2)^{\frac{1}{2}} \right|^2 \leq c_1 \mathbb{E}^j |\nabla_i f|^2 + \frac{J^2 c_1}{2^2} A_i^j(f)$$

*Proof.* We have

$$\begin{aligned} \left| \nabla_i (\mathbb{E}^j f^2)^{\frac{1}{2}} \right|^2 &= \left| \frac{1}{2} (\mathbb{E}^j f^2)^{\frac{1}{2}-1} \nabla_i (\mathbb{E}^j f^2) \right|^2 = \\ (4.1) \quad &\frac{1}{2^2} (\mathbb{E}^j f^2)^{-1} |\nabla_i (\mathbb{E}^j f^2)|^2 \end{aligned}$$

But from relationship (3.18) of Lemma 4.5, for  $\rho_j$  being the density of  $\mathbb{E}^j$  we have

$$\begin{aligned} |\nabla_i (\mathbb{E}^j f^2)|^2 &= |\nabla_i (\int \rho_j f^2 dx_j)|^2 \leq \\ (4.2) \quad &2^2 \left| \int \nabla_i (f^2) \rho_j dx_j \right|^2 + 2^2 \left| \int f^2 (\nabla_i \rho_j) dx_j \right|^2 \end{aligned}$$

For the second term in (4.2) we have

$$(4.3) \quad \left| \int f^2 (\nabla_i \rho_j) dx_j \right|^2 \leq J^2 |\mathbb{E}^j(f^2; \nabla_i V(x_j, x_i))|^2$$

While for the first term of (4.2) the following bound holds

$$\begin{aligned} \left| \int \nabla_i (f^2) \rho_j dx_j \right|^2 &= 2^2 |\mathbb{E}^j(f(\nabla_i f))|^2 \\ &\leq 2^2 (\mathbb{E}^j f^2) (\mathbb{E}^j |\nabla_i f|^2) \\ (4.4) \quad &= 2^2 (\mathbb{E}^j f^2) (\mathbb{E}^j |\nabla_i f|^2) \end{aligned}$$

where above we used the Hölder inequality. If we plug (4.3) and (4.4) in (4.2) we get

$$\begin{aligned} |\nabla_i (\mathbb{E}^j f^2)|^2 &\leq 2^2 2^2 (\mathbb{E}^j f^2) (\mathbb{E}^j |\nabla_i f|^2) + \\ &2^2 J^2 |\mathbb{E}^j(f^2; \nabla_i V(x_j, x_i))|^2 \end{aligned}$$

From the last relationship and (4.1) the lemma follows.  $\square$

The next lemma presents an estimates of  $A_i^j(f)$ .

**Lemma 4.2.** *Suppose that hypothesis (H0) and (H1) are satisfied. Then for  $i \sim j$  the following hold for some constant  $m_0 > 0$*

$$\nu A_i^j(f) \leq m_0 \nu |\nabla_j f|^2 + m_0 \sum_{t \sim j} \nu |\nabla_t f|^2$$

*Proof.* We can initially bound  $A^j(i)$  with the use of the following lemma.

**Lemma 4.3.** *For any probability measure  $\mu$  the following inequality holds*

$$\mu(|f|^2; h) \leq c_0 (\mu|f|^2)^{\frac{1}{2}} (\mu(|f - \mathbb{E}f|^2(|h|^2 + \mu|h|^2)))^{\frac{1}{2}}$$

for some constant  $c_0$  uniformly on the boundary conditions.

The proof of Lemma 4.3 can be found in [Pa2]. We then get

$$A_i^j(f) = (\mathbb{E}^j f^2)^{-1} |\mathbb{E}^j(f^2; W_i^j)|^2 \leq c_0^2 \mathbb{E}^j(|f - \mathbb{E}^j f|^2 U_i^j)$$

On the right hand side we can use the following entropic inequality (see [D-S])

$$(4.5) \quad \forall t > 0, \mu(uv) \leq \frac{1}{t} \log(\mu(e^{tu})) + \frac{1}{t} \mu(v \log v)$$

for any probability measure  $\mu$  and  $v \geq 0$ ,  $\mu v = 1$ . That will give

$$\nu A_i^j(f) \leq \frac{c_0^2}{\epsilon} \nu \mathbb{E}^j |f - \mathbb{E}^j f|^2 \log \frac{|f - \mathbb{E}^j f|^2}{\mathbb{E}^j |f - \mathbb{E}^j f|^2} + \frac{c_0^2}{\epsilon} \nu \mathbb{E}^j |f - \mathbb{E}^j f|^2 \log \mathbb{E}^j e^{\epsilon U_i^j}$$

If we use hypothesis (H0) for  $\mathbb{E}^j$  then we can bound  $A_i^k(f)$  by

$$(4.6) \quad \nu A_i^j(f) \leq \frac{c_0^2 c}{\epsilon} \nu \mathbb{E}^j |\nabla_j f|^2 + \frac{c_0^2}{\epsilon} \nu \mathbb{E}^j |f - \mathbb{E}^j f|^2 \log \mathbb{E}^j e^{\epsilon U_i^j}$$

For the exponent on the right and side of (4.6), we can compute

$$\begin{aligned} \log \mathbb{E}^j e^{\epsilon U_i^j} &= \log \left( e^{\epsilon \mathbb{E}^j |W_i^j|^2} \mathbb{E}^j e^{\epsilon |W_i^j|^2} \right) = \epsilon \mathbb{E}^j |W_i^j|^2 + \log \mathbb{E}^j e^{\epsilon |W_i^j|^2} \leq \\ &= \epsilon k + \epsilon k \sum_{i \sim j} d^s(\omega_i) + \log e^k e^{k \sum_{i \sim j} d^s(\omega_j)} = (\epsilon + 1)k + (\epsilon + 1)k \sum_{i \sim j} d^s(\omega_i) \end{aligned}$$

where above we used (H1.3) for some  $\epsilon > 0$  sufficiently small. Plugging the last one in (4.6) gives

$$\begin{aligned} \nu A_i^j(f) &\leq \frac{c_0^2 c}{\epsilon} \nu \mathbb{E}^j |\nabla_j f|^2 + \frac{(\epsilon + 1)k c_0^2}{\epsilon} \nu \mathbb{E}^j |f - \mathbb{E}^j f|^2 + \\ &\quad \frac{(\epsilon + 1)k c_0^2}{\epsilon} \sum_{i \sim j} \nu \mathbb{E}^j |f - \mathbb{E}^j f|^2 d^s(\omega_i) \leq \\ (4.7) \quad &\frac{(k\epsilon + k + 1)cc_0^2}{\epsilon} \nu |\nabla_j f|^2 + \frac{(\epsilon + 1)kc_0^2}{\epsilon} \sum_{i \sim j} \nu \mathbb{E}^i (|f - \mathbb{E}^j f|^2 d^s(\omega_i)) \end{aligned}$$

where above the second term on the rhs was bounded with the use of the Spectral Gap inequality for the  $\mathbb{E}^i$  measure. We can bound the last term on the right hand side of (4.7) with the use of Corollary 3.4.

$$\nu A_i^j(f) \leq \frac{((k\epsilon + k + 1)c + 4D_3(\epsilon + 1)k)c_0^2}{\epsilon} \nu |\nabla_j f|^2 + \frac{D_3(\epsilon + 1)kc_0^2}{\epsilon} \sum_{i \sim j} \nu |\nabla_i f|^2$$

□

If we combine together Lemma 4.1 and 4.2 we obtain the following corollary.

**Corollary 4.4.** *Suppose that hypothesis (H0) and (H1) are satisfied. Then for every  $i \sim j$  the following inequality holds*

$$\nu \left| \nabla_i (\mathbb{E}^j |f|^2)^{\frac{1}{2}} \right|^2 \leq G_1 \nu |\nabla_i f|^2 + G_2 \nu |\nabla_j f|^2 + G_2 \sum_{t \sim j, t \neq i} \nu |\nabla_t f|^2$$

for  $G_1 > 0$  and  $G_2 \leq 1$ .

We can now prove Proposition 2.6.

**Proof of Proposition 2.6.**

If we denote  $\{\sim i\} = \{i_1, i_2, i_3, i_4\}$  the neighbours of node  $i$ , then we can write

$$(4.8) \quad \nu \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} f^2)^{\frac{1}{2}} \right|^2 = \sum_{i \in \Gamma_1} \nu \left| \nabla_i (\mathbb{E}^{\Gamma_0} f^2)^{\frac{1}{2}} \right|^2 \leq \sum_{i \in \Gamma_1} \nu \left| \nabla_i (\mathbb{E}^{\{\sim i\}} f^2)^{\frac{1}{2}} \right|^2$$

We can bound the last one with the use of Corollary 4.4

$$(4.9) \quad \nu \left| \nabla_i (\mathbb{E}^{\{\sim i\}} f^2)^{\frac{1}{2}} \right|^2 = \nu \left| \nabla_i (\mathbb{E}^{i_1} \mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 \leq G_1 \nu \left| \nabla_i (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 + G_2 \nu \left| \nabla_{i_1} (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 + G_2 \sum_{t \sim i_1, t \neq i} \nu \left| \nabla_t (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2$$

Since  $i_1$  has distance bigger than the interaction one from  $i_2, i_3, i_4$ , the second term on the right hand side is

$$(4.10) \quad \nu \left| \nabla_{i_1} (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 = \nu |\nabla_{i_1} f|^2$$

For the third term, we can distinguish between the  $t$ 's in  $\{t \sim i_1, t \neq i\}$  which neighbour only  $i_1$  of all the nodes in  $\{\sim i\}$ , which we will denote  $i'_1$ , and the  $t$ 's in  $\{t \sim i_1, t \neq i\}$  which also neighbour another node in  $\{\sim i\}$ , which we will denote  $i_{1k}$  if  $i_k$  is the second node in  $\{\sim i\}$  that they neighbour. In that way, if the neighbours of  $i$ ,  $\{\sim i\} = i_1, i_2, i_3, i_4$  are numbered clockwise, then  $i_{12}$  neighbours both  $i_1, i_2$  and  $i_{14}$  neighbours both  $i_1, i_4$ . If we use again Corollary 4.4 to bound the third term on (4.9) then for the  $t$  in  $\{t \sim i_1, t \neq i\} = \{i'_1, i_{12}, i_{14}\}$  we have

$$(4.11) \quad \nu \left| \nabla_{i'_1} (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 = \nu |\nabla_{i'_1} f|^2$$

and

$$\begin{aligned}
\nu \left| \nabla_{i_{12}} (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 &= \nu \left| \nabla_{i_{12}} (\mathbb{E}^{i_2} \mathbb{E}^{\{i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 \leq \\
&G_1 \nu \left| \nabla_{i_{12}} (\mathbb{E}^{\{i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 + G_2 \nu \left| \nabla_{i_2} (\mathbb{E}^{\{i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 + \\
&G_2 \sum_{t \sim i_2, t \neq i_{12}} \nu \left| \nabla_t (\mathbb{E}^{\{i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 \leq G_1 \nu |\nabla_{i_{12}} f|^2 + \\
&G_2 \nu |\nabla_{i_2} f|^2 + G_2 \sum_{t \sim i_2, t \neq i_{12}} \nu \left| \nabla_t (\mathbb{E}^{\{i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2
\end{aligned}$$

Since  $\{t \sim i_2, t \neq i_{12}\} = \{i, i_{23}, i'_2\}$  the last becomes

$$\begin{aligned}
\nu \left| \nabla_{i_{12}} (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 &\leq G_1 \nu |\nabla_{i_{12}} f|^2 + G_2 \nu |\nabla_{i_2} f|^2 + G_2 \nu |\nabla_{i'_2} f|^2 + \\
&G_2 \nu \left| \nabla_i (\mathbb{E}^{\{i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 + G_2 G_1 \nu |\nabla_{i_2} f|^2 + \\
(4.12) \quad &G_2^2 \nu |\nabla_{i_3} f|^2 + G_2^2 \sum_{t \sim i_3, t \neq i_{23}} \nu |\nabla_t f|^2
\end{aligned}$$

where above was made use once more of Corollary 4.4. Similarly we can compute

$$\begin{aligned}
\nu \left| \nabla_{i_{14}} (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 &\leq G_1 \nu |\nabla_{i_{14}} f|^2 + G_2 \nu |\nabla_{i_4} f|^2 + G_2 \nu |\nabla_{i'_4} f|^2 + \\
&G_2 \nu \left| \nabla_i (\mathbb{E}^{\{i_2, i_3\}} f^2)^{\frac{1}{2}} \right|^2 + G_2 G_1 \nu |\nabla_{i_4} f|^2 + \\
(4.13) \quad &G_2^2 \nu |\nabla_{i_4} f|^2 + G_2^2 \sum_{t \sim i_4, t \neq i_{34}} \nu |\nabla_t f|^2
\end{aligned}$$

If we plug (4.10), (4.11), (4.12) and (4.13) in (4.9) we get

$$\begin{aligned}
\nu \left| \nabla_i (\mathbb{E}^{\{i_1, i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 &\leq G_1 \nu \left| \nabla_i (\mathbb{E}^{\{i_2, i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 + G_2 \nu \left| \nabla_i (\mathbb{E}^{\{i_3, i_4\}} f^2)^{\frac{1}{2}} \right|^2 + \\
&G_2 \nu \left| \nabla_i (\mathbb{E}^{\{i_2, i_3\}} f^2)^{\frac{1}{2}} \right|^2 + G_2 \check{A} \nu |\nabla_i f|^2 + \\
&G_2 \check{A} \sum_{j \sim i} \nu |\nabla_j f|^2 + G_2 \check{A} \sum_{k=1}^4 \nu |\nabla_{i'_k} f|^2 + \\
(4.14) \quad &G_2 \check{A} (\nu |\nabla_{i_{12}} f|^2 + \nu |\nabla_{i_{14}} f|^2 + \nu |\nabla_{i_{34}} f|^2)
\end{aligned}$$

for some positive constant  $\check{A} > 0$ . If we repeat the same calculation for the first three terms on the right hand side of the last inequality we will finally obtain

$$\nu \left| \nabla_i (\mathbb{E}^{\{\sim j\}} f^2)^{\frac{1}{2}} \right|^2 \leq G_1 \nu |\nabla_i f|^q + G_2 \check{B} \sum_{j \sim i} \nu |\nabla_j f|^2 + G_2 \check{B} \sum_{j: \text{dist}(j, i)=2} \nu |\nabla_j f|^2$$

for some constant  $\check{B} > 0$ . From the last one and (4.8) we finally have

$$\nu \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} f^2)^{\frac{1}{2}} \right|^2 \leq (G_1 + 8G_2\check{B})\nu |\nabla_{\Gamma_1} f|^2 + 4G_2\check{B}\nu |\nabla_{\Gamma_0} f|^2$$

which gives the proposition for sufficiently small  $J$  such that  $4G_2\check{B} < 1$ .

## 5. CONCLUSION

In the present work, we have looked in the Logarithmic Sobolev inequality for the infinite volume Gibbs measure. As explained in the introduction, the criterion presented in Theorem 2.1 can in particular be applied in the case of local specifications  $\{\mathbb{E}^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}^d, \omega \in \Omega}$  with no quadratic interactions for which

$$\|\nabla_i \nabla_j V(x_i, x_j)\|_{\infty} = \infty$$

Thus, we have shown that our results can go beyond the usual uniform boundness of the second derivative of the interactions considered in [Z1], [Z2], [M], [Pa2] and [O-R].

The main assumption about the local specification has been that the one site measure  $\mathbb{E}^{i, \omega}$  satisfies the Log-Sobolev Inequality with a constant uniformly to the boundary conditions. Then, under the assumption that the phase  $\phi(x) \geq d^p(x)$  for  $p \geq 2$  dominates over the interactions, in the sense that

$$|\nabla_j V(x_i, \omega_j)|^2 \leq k + k(d^s(x_i) + d^s(\omega_j))$$

for  $s \leq p - 1$ , the Log-Sobolev inequality is extended to the infinite dimensional Gibbs measure.

In addition to the two main natural conditions mentioned above, one further was placed on the interactions, that is (H1.1) that requires

$$\partial_{d(x)} V(x, y) \geq 0$$

One can look in extended the current result to the more general case where (H1.1) is not required, which will allow for a greater variety of non quadratic interactions to be considered.

## REFERENCES

- [A-B-C] C. ANÉ, S. BLACHÈRE, D. CHAFAÏ, P. FOUGÈRES, I. GENTIL, F. MALRIEU, C. ROBERTO, G. SCHEFFER, *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses. Soc. Math, 10, France, Paris (2000).
- [B] D. BAKRY, *L'hypercontractivité et son utilisation en théorie des semigroups*, Séminaire de Probabilités XIX, Lecture Notes in Math., 1581, Springer, New York, 1-144 (1994).
- [B-E] D. BAKRY AND M. EMERY, *Difusions hypercontractives*, Séminaire de Probabilités XIX, Springer Lecture Notes in Math. 1123, 177-206 (1985).
- [B-HK] J. BELLISARD AND R. HOEGN-KROHN, Compactness and the maximal Gibbs state for random fields on the Lattice, Commun. Math. Phys. 84, 297-327 (1982).
- [B-G] S.G. BOBKOV AND F. GOTZE, *Exponential integrability and transportation cost related to logarithmic sobolev inequalities*, J of Funct Analysis 163 1-28 (1999).

- [B-L] S.G. BOBKOV AND M. LEDOUX, *From Brunn-Minkowski to Brascamp-Lieb and to Logarithmic Sobolev Inequalities*, Geom. funct. anal. 10, 1028-1052 (2000).
- [B-Z] S.G. BOBKOV AND B. ZEGARLINSKI, *Entropy Bounds and Isoperimetry*. Memoirs of the American Mathematical Society, Vol: 176, 1 - 69 (2005).
- [B-H] T. BODINEAU AND B. HELFFER, *Log-Sobolev inequality for unbounded spin systems*, J of Funct Analysis 166, 168-178 (1999).
- [D-S] J.D. DEUSCHEL AND D. STROOCK, *Large Deviations*, Academic Press, San Diego (1989).
- [D] R. L. DOBRUSHIN, *The problem of uniqueness of a Gibbs random field and the problem of phase transition*, Funct. Anal. Appl. 2, 302-312 (1968).
- [G-R] I. GENTIL AND C. ROBERTO, *Spectral Gaps for Spin Systems: Some Non-convex Phase Examples*, J. Func. Anal., 180, 66-84 (2001).
- [G] L. GROSS, *Logarithmic Sobolev inequalities*, Am. J. Math. 97, 1061-1083 (1976).
- [G-Z] A. GUIONNET AND B. ZEGARLINSKI, *Lectures on Logarithmic Sobolev Inequalities*, IHP Course 98, pp 1-134 in Seminaire de Probabilite XXVI, Lecture Notes in Mathematics 1801, Springer (2003).
- [H-Z] W. HEBISCH AND B. ZEGARLINSKI, *Coercive inequalities on metric measure spaces*. J. Func. Anal., 258, 814-851 (2010).
- [I-P] J. INGLIS AND I. PAPAGEORGIOU, *Logarithmic Sobolev Inequalities for Infinite Dimensional Hörmander Type Generators on the Heisenberg Group*. Potential Anal., 31, 79-102 (2009).
- [L] M. LEDOUX, *Concentration of measure and logarithmic Sobolev inequalities*, Seminaire de Probabilites, XXXIII, Lecture Notes in Math. 1709, Springer-Verlag, pp. 120-216 (1999).
- [M] K. MARTON, *Logarithmic Sobolev Inequality for Weakly Dependent Random Variables*. (preprint)
- [O-R] F. OTTO AND M. REZNIKOFF, *A new criterion for the Logarithmic Sobolev Inequality and two Applications*, J. Func. Anal., 243, 121-157 (2007).
- [Pa1] I. PAPAGEORGIOU, *A Note on the Modified Log-Sobolev Inequality*, Potential Anal., 35, 275-286 (2011).
- [Pa2] I. PAPAGEORGIOU, *The Logarithmic Sobolev Inequality in Infinite dimensions for Unbounded Spin Systems on the Lattice with non Quadratic Interactions*, Markov Proc. Related Fields, 16, 447-484 (2010).
- [Pr] C.J. PRESTON, *Random Fields*, LNM 534, Springer (1976).
- [R-Z] C. ROBERTO AND B. ZEGARLINSKI, *Orlicz-Sobolev inequalities for sub-Gaussian measures and ergodicity of Markov semi-groups*, J. Func. Anal., 243 (1), 28-66 (2007).
- [Y] N. YOSHIDA, *The log-Sobolev inequality for weakly coupled lattice field*, Probab. Theor. Relat. Fields 115, 1-40 (1999).
- [Z1] B. ZEGARLINSKI, *On log-Sobolev Inequalities for Infinite Lattice Systems*, Lett. Math. Phys. 20, 173-182 (1990).
- [Z2] B. ZEGARLINSKI, *The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice*, Comm. Math. Phys. 175, 401-432 (1996).